

Visual Analysis of Spatial Variability and Global Correlations in Ensembles of Iso-Contours

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Supplemental Material

Integration of MVN distribution

Here, we show a proof for the identity between Eq. 4 and Eq. 5 in the paper.

Let $\mathcal{N}(\mathbf{x}, \mu, \Sigma)$ denote the multivariate normal (MVN) density with mean $\mu \in \mathbb{R}^r$ and covariance matrix $\Sigma \in \mathbb{R}^{r \times r}$:

$$\mathcal{N}(\mathbf{x}, \mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)\Sigma^{-1}(\mathbf{x} - \mu)\right)$$

We are interested in integrating \mathcal{N} over an intersection of two slabs (= regions between two parallel hyper-planes):

$$\begin{aligned} \hat{\mathcal{S}}_a &:= \{\mathbf{x} \in \mathbb{R}^r \mid |\mathbf{a}^T \mathbf{x} + c| \leq s\} \\ \hat{\mathcal{S}}_b &:= \{\mathbf{x} \in \mathbb{R}^r \mid |\mathbf{b}^T \mathbf{x} + d| \leq t\} \end{aligned} \quad (1)$$

Here, we assume $\mathbf{a}, \mathbf{b} \in \mathbb{R}^r$ and $c, d, s, t \in \mathbb{R}$ to be fixed parameters. Note that, for simplicity reasons, $\hat{\mathcal{S}}$ is defined slightly different to \mathcal{S} in the paper (we use $\mathbf{a} = U^T \mathbf{w}_y$, $\mathbf{b} = U^T \mathbf{w}_z$, $c = \bar{\mathbf{d}}^T \mathbf{w}_z$ and $d = \bar{\mathbf{d}}^T \mathbf{w}_z$ here).

For the integration, we will perform a change of variables which aligns \mathbf{a} and \mathbf{b} with the first and second coordinate axis, respectively, allowing us to collapse the other dimensions of the MVN distribution. Therefore, we define the affine transformation $q(\mathbf{y}) = T\mathbf{y} + \mu$, where the $r \times r$ matrix $T := (S^{-1})^T$ is given through $S := (\mathbf{a}, \mathbf{b}, \mathbf{t}_3, \dots, \mathbf{t}_r)$. The columns $\mathbf{t}_i \in \mathbb{R}^r$ can be chosen arbitrarily, such that S has full rank and $\det(S) > 0$. Let \mathbf{e}_i denote the i -th unit vector, then, by construction, the following holds:

$$S\mathbf{e}_1 = \mathbf{a} \wedge S\mathbf{e}_2 = \mathbf{b} \quad \Rightarrow \quad T^T \mathbf{a} = \mathbf{e}_1 \wedge T^T \mathbf{b} = \mathbf{e}_2 \quad (2)$$

To perform the change of variables, we also need to apply q^{-1} to the integration regions $\hat{\mathcal{S}}(\mathbf{a}, c)$ and $\hat{\mathcal{S}}(\mathbf{b}, d)$, which can be achieved by substituting $\mathbf{x} = T\mathbf{y} + \mu$ in Eq. (1):

$$\begin{aligned} \hat{\mathcal{S}}'_a &:= q^{-1}(\hat{\mathcal{S}}_a) = \{\mathbf{y} \in \mathbb{R}^r \mid |\mathbf{a}^T T\mathbf{y} + \mathbf{a}^T \mu + c| \leq s\} \\ \hat{\mathcal{S}}'_b &:= q^{-1}(\hat{\mathcal{S}}_b) = \{\mathbf{y} \in \mathbb{R}^r \mid |\mathbf{b}^T T\mathbf{y} + \mathbf{b}^T \mu + d| \leq t\} \end{aligned}$$

Using Eq. (2), we can simplify this to

$$\begin{aligned} \hat{\mathcal{S}}'_a &= \{\mathbf{y} \in \mathbb{R}^r \mid |(\mathbf{y})_1 + \mathbf{a}^T \mu + c| \leq s\} \\ \hat{\mathcal{S}}'_b &= \{\mathbf{y} \in \mathbb{R}^r \mid |(\mathbf{y})_2 + \mathbf{b}^T \mu + d| \leq t\}. \end{aligned} \quad (3)$$

Finally, the integral which we are interested in is:

$$\begin{aligned} &\int_{\hat{\mathcal{S}}_a \cap \hat{\mathcal{S}}_b} \mathcal{N}(\mathbf{x}, \mu, \Sigma) d\mathbf{x} \\ &= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \int_{\hat{\mathcal{S}}_a \cap \hat{\mathcal{S}}_b} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1}(\mathbf{x} - \mu)\right) d\mathbf{x} \end{aligned}$$

Using the previously defined transformation $q(\mathbf{y})$, we substitute $\mathbf{x} = T\mathbf{y} + \mu$:

$$\begin{aligned} &= \frac{\det(T)}{\sqrt{(2\pi)^n \det(\Sigma)}} \int_{\hat{\mathcal{S}}'_a \cap \hat{\mathcal{S}}'_b} \exp\left(-\frac{1}{2}(T\mathbf{y})^T \Sigma^{-1}(T\mathbf{y})\right) d\mathbf{y} \\ &= \frac{1}{\sqrt{(2\pi)^n \det(T^{-1}\Sigma(T^T)^{-1})}} \\ &\quad \cdot \int_{\hat{\mathcal{S}}'_a \cap \hat{\mathcal{S}}'_b} \exp\left(-\frac{1}{2}\mathbf{y}^T (T^T \Sigma^{-1} T) \mathbf{y}\right) d\mathbf{y} \\ &= \int_{\hat{\mathcal{S}}'_a \cap \hat{\mathcal{S}}'_b} \mathcal{N}(\mathbf{y}, 0, T^{-1}\Sigma(T^T)^{-1}) d\mathbf{y} \\ &= \int_{\hat{\mathcal{S}}'_a \cap \hat{\mathcal{S}}'_b} \mathcal{N}(\mathbf{y}, 0, S^T \Sigma S) d\mathbf{y} \end{aligned}$$

The integration region is now bounded by hyper-planes which are orthogonal either to the first or to the second coordinate axis (see Eq. (3)). Hence, regardless of the original dimensionality r , we can collapse all remaining dimensions of the MVN distribution, which gives the two-dimensional integral

$$= \int_{-s-\mathbf{a}^T \mu - c}^{s-\mathbf{a}^T \mu - c} \int_{-t-\mathbf{b}^T \mu - d}^{t-\mathbf{b}^T \mu - d} \mathcal{N}((y_1, y_2)^T, 0, \Sigma') dy_2 dy_1,$$

with integrations bounds according to Eq. (3) and with $\Sigma' \in \mathbb{R}^{2 \times 2}$ being the upper left 2×2 submatrix of $S^T \Sigma S$:

$$\Sigma' = \begin{pmatrix} \mathbf{a}^T \Sigma \mathbf{a} & \mathbf{a}^T \Sigma \mathbf{b} \\ \mathbf{a}^T \Sigma \mathbf{b} & \mathbf{b}^T \Sigma \mathbf{b} \end{pmatrix}.$$

This is the expression we were looking for. Additionally, we can transform it into a normalized ρ -form by rescaling along both coordinate axes. This is the format that is typically accepted by libraries which can calculate 2D rectangular normal probabilities:

$$\int_{\mathcal{S}_a \cap \mathcal{S}_b} \mathcal{N}(\mathbf{x}, \mu, \Sigma) d\mathbf{x}$$

$$= \int_{\frac{s-\mathbf{a}^T \mu - c}{\mathbf{a}^T \Sigma \mathbf{a}}}^{\frac{s-\mathbf{a}^T \mu - c}{\mathbf{a}^T \Sigma \mathbf{a}}} \int_{\frac{t-\mathbf{b}^T \mu - d}{\mathbf{b}^T \Sigma \mathbf{b}}}^{\frac{t-\mathbf{b}^T \mu - d}{\mathbf{b}^T \Sigma \mathbf{b}}} \mathcal{N}((y_1, y_2)^T, \mathbf{0}, \Sigma'') dy_2 dy_1,$$

$$\Sigma'' = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{with} \quad \rho = \frac{\mathbf{a}^T \Sigma \mathbf{b}}{\sqrt{(\mathbf{a}^T \Sigma \mathbf{a}) \cdot (\mathbf{b}^T \Sigma \mathbf{b})}}.$$